DMT INSTITUTE

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DEVELOPING MATHEMATICAL THINKING FRAMEWORK AND CLASSROOM STRUCTURE

Students at the elementary and middle school levels are not performing well in mathematics, and their overall performance does not improve over time (NCES, 2013, 2015). Based on recent reports, these students are not considered mathematically literate or proficient when compared with national standards (Lemke et al., 2004; Mullis, Martin, Gonzalez, & Chrostowski, 2004; NMAP, 2008; NRC,2001). Several researchers have demonstrated that students who complete kindergarten with inadequate knowledge of basic mathematical concepts and skills continue to experience difficulties with mathematics throughout their elementary and secondary years (Duncan et al., 2007; Jordan, Kaplan, Ramineni, & Locuniak, 2009; Morgan, Farkas, & Qiong, 2009). By fourth grade, student performance is deemed poor based on various measures of mathematics (Clements, Xiufeng, & Sarama, 2008; D. Clements & Sarama, 2007; Dossey, 1992; Gersten et al., 2009; NRC, 2009; Reese,Miller, Mazzeo, & Dossey, 1997). For instance, Mitchell and colleagues describe students at the fourth-grade level as consistently performing poorly on assessment items, even items of low complexity, showing no improvement by the eighth and twelfth grades (Dossey, 1992; Mitchell, Hawkins, Stancavage, & Dossey, 2000). These studies point to the change needed in how students, particularly those at the elementary level, are taught (Chernoff, Flanagan, McPhee, & Park, 2007; Ginsburg, Lee, & Boyd, 2008).

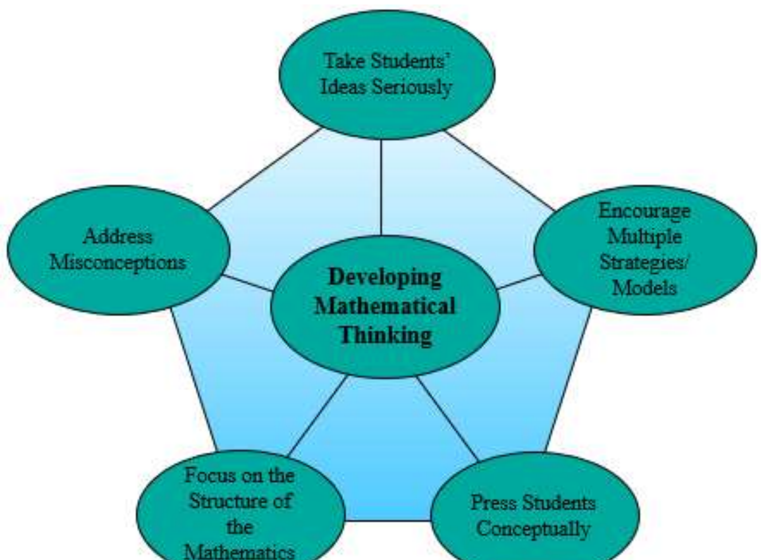
There are interrelated mechanisms for improving student achievement: professional development, instruction, and curricular materials (Desimone, 2009; Desimone, Porter, Garet, Yoon, & Birman, 2002). Over the past three decades, much work has been pushing instruction and materials tobe more in line with a socio-constructivist approach to teaching and learning (NCTM, 1989, 1991). This teaching approach is often referred to as an inquiry approach in the literature, while curricular materials are referred to as standards-based or reformed mathematics. Through the 1990s, the National Science Foundation (NSF) funded the development of numerous K-12 programs (Senk & Thompson, 2003). These curricular materials (e.g., textbooks) focus on building conceptual knowledge through rich contexts (Trafton, Reys, & Wasman, 2001). In addition, much of the research has supported a culturally based education (CBE) approach for Native American students (Demmert & Towner, 2003; Lipka & Adams, 2004).

There is evidence that teaching through a socio-constructivist approach positively affectsteachers and students. Cognitively Guided Instruction (CGI) is one approach that has demonstrated changes in teachers' beliefs and knowledge on how mathematics should be taught, as well as in improved student learning (Carpenter, Fennema, Peterson, Chiange, & Loef, 1989; Carpenter, Franke, Jacobs, & Fennema, 1998; Fennema et al., 1996; Hiebert et al., 1996). This type of teaching also develops positive attitudes and beliefs in students (Wood & Sellers, 1997), increases students' participation in problem-solving (Weber, Radu, Mueller, Powell, & Maher, 2010), and enhances the involvement of all kinds of students (Sullivan, Mousley, & Zevenberger, 2006).

The following section describes the theoretical foundation for Developing Mathematical Thinking's (DMT) pedagogical framework. The DMT's framework draws much from the socio-constructivist perspective; nevertheless, it is unique as a pedagogical framework. It has additional components drawn from other views of learning that help enhance the basic tenetsof socio-constructivism. We have also described a framework for how curricular modules that address each dimension should be built. Finally, we have added a pilot study that evaluated the effectiveness of a sample module.

THEORETICAL FRAMEWORK FOR INSTRUCTION – DEVELOPING MATHEMATICAL THINKING

One critical role of a teacher is to create equitable learning conditions that foster understanding so students can become better problem-solvers in and outside of school (Hiebert et al., 1996; Newmann& Associates, 1996). To do this, we propose to examine an instructional framework, Developing Mathematical Thinking—DMT (Brendefur, Thiede, Strother, Bunning, & Peck, 2013; Brendefur, Thiede, Strother, Jesse, & Sutton, 2016), which focuses on five critical dimensions: (a) taking students' ideas seriously, (b) pressing students conceptually, (c) encouraging multiple strategies and models, (d) addressing misconceptions, and (e) focusing on the structure of mathematics. These five dimensions frame an approach to teaching mathematics for a better understanding (Carpenter & Lehrer, 1999), in addition to incorporating notions of "progressive formalization" and "mathematizing" (Freudenthal, 1973, 1991; Treffers, 1987).



As Gravemeijer andvan Galen (2003) describe, progressive formalization is a process of first allowing students to develop informal strategies for solving contextual problems and ways to model these approaches, and then, by critically examining both these strategies and models, teachers press students to develop more sophisticated, formal, conventional, and abstract strategies and algorithms. By comparing solution strategies and examining the relationship among enactive, iconic, and symbolic models (Bruner, 1964, 1996), students learn which manipulations make sense for given contexts and are encouraged to develop more generalizable procedures. Although rarely would you observe any of these dimensions or instructional practices exclusively, we examined each dimension individually for theoretical constructs and situations of practice – both instructional and for the curriculum.

TAKING STUDENTS' IDEAS SERIOUSLY (TSIS)

Ideas are taken seriously when students are challenged to solve a novel but meaningful mathematics task, allowed to share their initial ideas, and encouraged to connect what they already know to other related mathematical concepts (Hiebert et al., 1996; Romberg & Kaput, 1999). There are several advantages to placing students in such situations wherein their intuitive understanding is addressed and, in some instances, confronted by the teacher. Students are placed – typically briefly – in a state of cognitive dissonance, which enables them to begin the process of perseverance and schema development (Driscoll, 2004). Tapping into prior knowledge is critical to engaging students in mathematics (Carpenter & Lehrer, 1999).

Two overarching structures frame teaching for a richer-deeper understanding and ensure continuous growth in student understanding: progressive formalization and mathematizing. TSIS is the starting point for the developing notions of horizontal and vertical mathematization (Treffers, 1987). Horizontal mathematization pertains to students' representing a contextualized problem mathematically to find a solution strategy. Vertical mathematization involves taking mathematics to a higher level as students make their representations (the second dimension) and strategies objects of mathematical examination (the third dimension). Mathematizing covers activities such as generalizing, justifying, formalizing, and curtailing – including, but not limited to, developing an abstract algorithm (Gravemeijer & van Galen, 2003). By emphasizing both types of mathematizing in classrooms, teachers must focus on the inherent structure (the fourth dimension) of the emerging mathematical ideas. In addition, teachers must address students' misconceptions (the fifth dimension) as they arise so that they do not hinder the progression of mathematizing. One outcome of mathematizing is that teachers connect students' informal ideas, many of which may be developed outside of school, with more formal mathematical ideas. An assumption made as part of the DMT process is that students' informal ideas, conceptions, and strategies anticipate learning more formal mathematics later in their classroom experience.

According to Hiebert and Carpenter, "one would predict that if children possessed internal networksconstructed both in and out of school, and if they recognized the connections between them, their understanding and performance in both settings would improve" (Hiebert & Carpenter, 1992, p. 72). Such a process starts with carefully chosen tasks – typically contextualized (Doerr, 2006; Larsen & Bartlo, 2009; M. Simon & Tzur, 2004). To solve these tasks, students must model the situation to some degree. Rather than beginning with standard algorithms and attempting to concretize them, teaching starts with students' common-sense solutions to contextual problems that are seemingly real. By reflecting on their procedures for solving, students develop and are introduced to more sophisticated models (2nd dimension) and procedures that can also be used in other situations (Gravemeijer & van Galen, 2003, p. 114).

Teachers should initially take students' ideas seriously by placing them in situations to activate their prior knowledge and extend their ideas. This is the beginning to develop mathematical understanding where teachers must attend to the process of progressive formalization and mathematizing. As Carpenter and Lehrer (1999) argue, "For learning with understanding to occur, instruction needs to provide students the opportunity to develop productive relationships, extend and apply their knowledge, reflect about their experiences, articulate what theyknow, and make knowledge their own" (p. 32).

Once students are placed in such a problem-solving situation with no initial guidance from the teacher, they share their ideas with the teacher, other students, or the entire class to expand their mathematical thinking. For too many elementary teachers, these student solution strategies and notations may seem inefficient or informal, but by eliciting and valuing students' initial solution strategies, teachers can connect students' thinking to more efficient and abstract methods

(Freudenthal, 1973, 1991; Gravemeijer & van Galen, 2003; Treffers, 1987). Unfortunately,many elementary teachers do not have the pedagogical or mathematical knowledge to equitably challenge all students (Ball, Hill, & Bass, 2005; Hill, Sleep, Lewis, & Ball, 2007) and create a reflective and instructional discourse (Brendefur & Frykholm, 2000). Therefore, curricular resourcesare needed to provide teachers with access to possible ways to differentially build on groups of students' informal and formal knowledge. This process is described in the section on the module framework.

ENCOURAGING MULTIPLE STRATEGIES AND MODELS (EMSM)

After students have had time for problem-solving and sharing their intuitive approaches, the next dimension is to encourage multiple strategies and models. Students must first be placed in situations that enable them to examine their approach to problem-solving and compare it with other approaches; second, they must be given opportunities to model the problem differently (Romberg & Kaput, 1999).

Modeling is a critical component in developing mathematical thinking. Knowledge originates from students' attempts to model contextual situations. The initial models eventually become the basis for solving related problems and a means of support for more formal mathematical reasoning (Gravemeijer & van Galen, 2003). As Cobb (2000) describes, this use of modeling "…implies a shift in classroom mathematical practices such that ways of symbolizing developed to initially express informal mathematical activity take on a life of their own and are used subsequently to support more formal mathematical activity in a range of situations" (p. 319). In this way, modeling is a fundamental process in learning mathematics. However, this view of models and modeling contrastswith the current instructional practices in mathematics in which models are used to "concretize expert knowledge" (Gravemeijer & van Galen, 2003, p. 118), such as when students are taught to model the traditional regrouping algorithm for subtraction with base-10 blocks. Likewise, contextual problems are traditionally presented only after students have mastered standard algorithmic ways of solving problems.

Progressive formalization is taking students' ways of modeling through enactive, iconic, and symbolic representations (Bruner, 1964) to become more formalized without making giant leaps. This process is addressed through both horizontal and vertical mathematizing. The focus is on students' ways of using models rather than on teacher-dictated ways. By enactingaspects of "progressive formalization" and "mathematizing," teachers develop a classroom practice based on the tenets of teaching for understanding.

Thus, this second dimension involves developing students' understanding of various models and approaches to solving problems (Dolk & Fosnot, 2006; NCTM, 2000; Romberg & Kaput, 1999). When students generate, evaluate, and utilize different mathematical strategies and models, they recognize the many ways to solve problems and represent solutions (Bruner, 1964). Various strategies and models highlight other aspects of mathematics, and by examining the same situation through different lenses, students' overall understanding of the topic deepens.

In the past, most reform or standards-based curricula have required students to solve a problem in two ways. However, students tend to represent the problem in two ways that are not distinct from each other and therefore do not address progressive formalization. In contrast, our module framework provides teachers with tasks that allow students to use, discuss, and practice with more informal and formal enactive, iconic, and symbolic models. Contexts are also carefully chosen to enable more logical connections between students' initial and informal modelsand the more mathematically valuable models. For example, if the area model is a desired iconic model, students would be given contextualized problems involving the covering of flat space, e.g., tiles on a floor or a map with gridlines used to find distances and areas of geographical regions (Leinwand & Ginsburg, 2007; Watanabe, 2015).

PRESSING STUDENTS CONCEPTUALLY (PSC)

The third dimension shifts instructional practices beyond merely procedural understanding to pressing students to conceptualize mathematics. Here, the focus is on building connections among mathematical strategies and models to progressively formalize students' ideas and methods for solving problems (Carpenter & Lehrer, 1999; NRC, 2001; Siegler & Alabali, 2004). For example, once students have had the chance to work on their solution methods, teachers must urge them to connect and compare methods, generalize new situations, and relate to formal mathematical terms and conventions. Through this process of connection and generalization, students' informal methods become more formal and efficient (Carpenter & Lehrer, 1999; Gravemeijer & van Galen, 2003).

Adopting a cognitive perspective on understanding, teachers aim to create classroom experiences that encourage students to *incorporate and organize* new information into a well-connected network of foundational mathematical ideas. For example, when initially studying multiplication, students

show a better understanding of when they can connect the operation to their previous (or concurrently) developed understandings of addition, patterning, and area.

From a social perspective, teachers should assign tasks and activities that place students in situations where they *reflect* on how they solve problems and *articulate* relationships among different strategies or concepts. Students demonstrate this type of understanding when, for example, explaining their problem-solving methods and analyzing those of others. By asking students togenerate and compare an iconic model to a symbolic model, they have the chance to understand the critical elements of mathematics. Here, to build a conceptual understanding, students are pressed to make connections between existing knowledge (informal ideas) and new knowledge (more formal mathematical ideas) (Hiebert & Carpenter, 1992). By critically examining their own and others' strategies and models, students build an understanding which exemplifies the importance of social interactions in classrooms. Hiebert and Carpenter (1992) state the following:

By thinking and talking about similarities and differences between arithmetic procedures, students can construct relationships. The instructional goal is not necessarily to inform oneprocedure by the other but rather to help students build a coherent mental network in which all pieces are joined to others with multiple links (Hiebert & Carpenter, 1992, p. 68).

This discourse around mathematical ideas supports all students, including struggling learners (Brendefur & Frykholm, 2000; Moschkovich, 1999, 2012). Through the interlinked processes of modeling situations mathematically and analyzing and comparing different methods, teachers press students to progressively formalize their ideas using more abstract mathematical ideas. For example, first graders might initially solve a problem using cubes and then be guided to represent the situation using a bar model or number line, and eventually, with symbolic notations. By asking students to connect several models and methods of their thinking with questions or problem variations, a teacher can enable students to progressively formalize their initial ideas.

FOCUSING ON THE STRUCTURE OF MATHEMATICS (FSM)

Focusing on structure allows students to understand and establish connections among fundamental concepts and particular topics being studied (NCTM, 2000; NGA, 2010). Structure, here, pertain to the elements of mathematics that remain constant across grade levels. For instance, the concepts of unit, composing, decomposing, iteration, partitioning, equivalence, and relationshipsare structural components for the concept of number. Understanding that the number 28 is composedof two units of size 10 and eight units of size one is necessary to understand place value, or that by partitioning one into 10 equivalent size units, you get a new unit of one-tenth, which has the relationship of iterating itself 10 times to become one. Therefore, focusing on structure helps students see how foundational ideas extend across grade levels and topics. By emphasizing connections across different topics, students need not be limited to memorized procedures for each particular case and can instead solve problems in related contexts.

Typically, teachers and students perceive mathematics as a series of procedures and definitions that increase complexity throughout the K-12 curriculum. Specific fundamental ideas or "structural components" appear continually throughout mathematics, whether in second or eleventh grade.

When instruction does not focus on the structure of mathematics, students often rely on memorized tricks or formulas, leading to difficulty in solving complex problems or applying mathematics to new situations. The module framework embeds the language of the structural components within each lesson in the task and formative assessment design, as well as in the examples of how studentsmight articulate and critique their own and others' mathematical models.

ADDRESSING MISCONCEPTIONS (AM)

The fifth dimension involves using students' mistakes and misconceptions as valuable tools to build mathematical understanding (Borasi, 1987, 1994). By focusing teaching practices on the first four dimensions of teaching mathematics, teachers shift their attention toward (a) students' informal strategies for solving problems, (b) the mathematical connections to and among multiple mathematical models and formal solution strategies, (c) developing a deep conceptual understanding, and (d) the structure of the mathematics, which lead to student misconceptions. By acknowledging and addressing them, teachers encourage students to make sense of and correct their flawed ways of thinking, rather than glossing over them or ignoring them altogether.

Mistakes often recur even after teachers demonstrate a correct procedure because they can stem from deeper mathematical misconceptions. By being aware of why and how misconceptions develop and taking the time to address them through models and discussion, teachers can subject students to a deeper understanding that precludes such mistakes. Thus, the module framework uses a modified version of Webb's (2002) depth of knowledge levels to introduce mistakes and misconceptions that other students might have. This presents relevant tasks each day as an integral part of doing mathematics and offers an opportunity for students to engage in justification, evaluation, and inquiry (Borasi, 1987).

SUMMARY OF THE FIVE DIMENSIONS OF DMT

The five key elements of DMT grow out of the concept that (a) mathematics comprises underlying, inherently related constructs and (b) students learn mathematics by creating web-like orhierarchical organizations for these constructs. With the introduction of the CCSS in mathematics, teachers and administrators have found that traditional textbooks do not adequately address their needs regarding the content, sequence of materials, levels of questioning, and discourse or differentiation (W.H. Schmidt, 2012; William H Schmidt & Burroughs, 2013). Many states, such asGeorgia and New York, have attempted to create curricular units to address these needs. Unfortunately, these materials, similar to the NSF-backed reformed curriculum of the 1990s, are different enough in terms of content, models, and presentation that teachers will struggleto implement them successfully (Obara & Sloan, 2010). Another issue is that these materials do nothelp teachers address the needs of marginalized students or build the adequate language needed to dialectically argue why a mathematical model works efficiently for specific situations (Doabler et al., 2012; Moschkovich, 2012). Therefore, a framework that addresses the five DMT components need to be developed and studied.

DMT INSTRUCTION AND FRACTION EXAMPLE

As previously described, math achievement can improve teaching for understanding. Still, often, it may not occur due to factors such as teachers' beliefs about mathematics and teaching, their knowledge of mathematics and mathematics pedagogy, students' difficulty with language-rich tasks, and students' lack of perseverance. Based on the five dimensions of DMT's instructional model, we first describe what instruction should be in a classroom and then create a framework to build curricular modules that incorporate progressive formalization and mathematizing. Because of the empirical evidence that teaching and learning fractions are difficult (Barnett-Clarke, Fisher, Marks, & Sharon, 2011; Jinfa Cai & Nie, 2007; Jinga Cai & Silver, 1995; Pitkethly & Hunting, 1996; Richland, Stigler, & Holyoak, 2012), we have highlighted instruction on the topic of fractionsfor third graders.

In an inquiry approach or reformed curriculum, a typical day includes a warmup that takes about two minutes, launching a problem-solving situation, student exploration, and a discussion (Stein, Engle, Smith, & Hughes, 2008). Several elements are missing, which we address: (a) ways to progressively formalize students' thinking, (b) modeling situations through context and enactive, iconic, and symbolic representations, (c) ways to formalize their initial, usually informal, language, (d) discussions that allow for critiquing and improving theirdiscourse, (e) conceptualizing the problem situations, and (f) focusing on the structure of mathematics.

Table 1. Overview of a DMT Instructional Lesson

	Focus	*DMT Connection*
Warmup		
Skill-Building	Start with a 2–5-minute warmup.	
Problem-Solving		

Problem-Solving Situation	Task Features Typically, a contextual problem or visual situation wherein a range of informal and formal modes of representation can be utilized to solve the problem. Presses students' cognitively.	TSIS: Enables mathematizing and progressive formalization througha task that provides multiple opportunitiesto examine informally and formal representational modes
Task Facilitation Structure		
Individual & Small Group Discussion	Students cognitively engage in the task at their level of mathematical understanding. Builds students' awareness about their knowledge and understanding of a topic Allows teachers time to find and, when needed, press participants individually for particular models	PSC & EMSM: Enables horizontal mathematizing through individuals' cognitive engagement with a contextual problem thatthey must represent mathematically
Whole-Group Discussion	Provides a pedagogical example of how a whole-class discussion should be led Presses understanding of and connections among multiple models to build knowledge	FSM, EMSM, & PSC: Presses horizontal mathematizing and progressive formalization through the examination ofmultiple solution methods with a facilitation focus on building connections among and progressively formalizing multiple ways of mathematically modeling the situation
Varied Tasks and Practice		
		EMSM:

Varied Tasks	Provides varied tasks with opportunities to build skills and understanding of different models Allows students to practice using the structure to articulate concepts	Increases horizontal mathematizing and ensures concepts are understood, and skills become efficient; mathematical languageis expanded
Varied Practice	Provides practice for the newly developed knowledge by asking students to practice with regard to contexts, iconic models, and symbolic representations	FSM: Provides an example ofhorizontal mathematizing by pressing students to connect contexts to models

MODULE OVERVIEW

Overall, each module highlights an overarching historical and cultural theme used to build the lessons. In our module, we have highlighted a blank theme. Then, as will be explained in more detailbelow, each lesson (which may take more than 1 class period) will include the following:

- *1-2 minutes of warmup*
- *Problem-solving situations*
- *Explanation of mathematics concepts and ideas (with historically, culturally relevant, andmathematically accurate ideas)*
- *Varied tasks (completed in small groups or individually)*
- *Varied Practice (enactive, iconic, and symbolic, or context, iconic, and symbolic)*
- *A review with different questions after every few lessons (skill, problem-solving, conceptual, and justification) will be incorporated as practice and a checkpoint for teachers.* We will highlight how the discussed module components play out in an instructional setting.

TASK CREATION OR SELECTION

The opening problem-solving task should (a) be accessible to students while being challenging, (b) allow for multiple ways to model or represent the problem situation, and (c) engage students in a key mathematical topic. The following task was used as the starting point for the module on fractions:

> *Courtney is making a glaze to apply to the ham. She needs 1/8 cup of coconut sugar and wants toknow whether it is the same as two 1/4 cups. Explain mathematically whether she is correct or not.*

Contexts incorporating misconceptions allow students the chance to explain their thinking by modeling with an enactive, iconic, or symbolic representation. This task is accessible and addresses equivalent fractions, a critical topic for third graders.

INDIVIDUAL AND SMALL GROUP WORK

This task is performed without any whole class "guidance" from the teacher. Students are allowed time to work individually and, after a few minutes, are prompted to work in small groups. Teachers are encouraged to group students who solved the task differently among themselves and then prompted to ask procedural and conceptual questions:

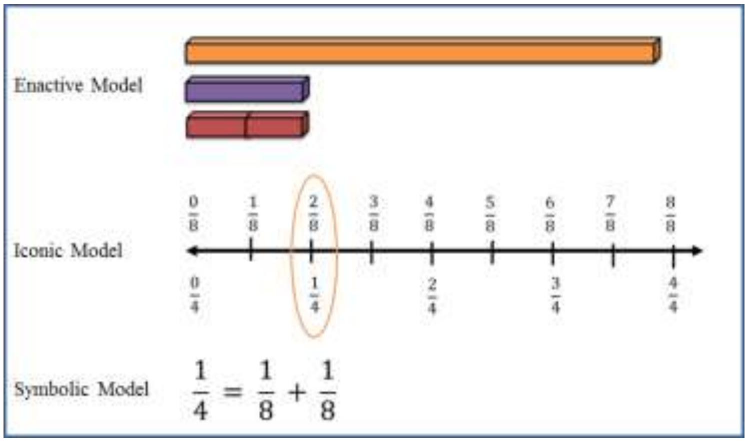
- For individuals struggling to figure out how problem-solving should be approached – *Is there a way to create or draw a diagram representing the situation?*
- For individuals who approached the problem with an "educated" guess and check strategy, or verbally – *Is there a way for you to notate your thinking to communicate it with others?*
- For individuals who solved the problem symbolically – *What would a visual diagram that matches your expression or equation look like?*
- For groups with multiple solution strategies within their group – *How are the various approaches related? Find an element of your problem-solving process in someone else's drawing or table.*

This part of task facilitation aims to engage students in a solution path for the task cognitively. This allows them to take ownership over the process of problem-solving and builds their explicit awareness of their knowledge and understanding of the topic. This makes them moreopen to establishing connections with other solution paths further into the module. In addition, it provides the critical perspective that anyone – no matter their knowledge – can understand and solve a problem when presented in a manner where multiple solution strategies, including informal methods, are both possible and encouraged. In short, this process addresses (a) students' ideas seriously, (b) progressive formalization to build a conceptual understanding, and (c) the initialstages of horizontal mathematizing.

WHOLE-GROUP DISCUSSION

In addition to posing probing questions for individuals and small groups during task facilitation, the teaching module includes possible participant-generated models for a whole-class discussion about the task. For this particular task, the module encourages teachers to find three types of models and ask three students to present their approaches. If teachers cannot find three model types, they can give examples on PowerPoint and ask students to describe what they are observing. Figure 2 highlights the criteria: (a) fraction rods (Jordan et al.), (b) number line, and (c)equations.

Figure 2: Enactive, Iconic, and Symbolic Models



If teachers find a model, which highlights an important idea or relationship, they are encouraged to ask students to recreate it on the board for the whole-class discussion. These models are generated during small group discussions. During a whole-class discussion, each student is asked to explain procedurally what they did to solve the problem. Then, the debate is opened up to the whole class to explain similarities and differences among the models and discuss where each model wouldwork most efficiently and where it would not.

For the task, possible questions are generated for the teacher to ask. These formative assessment questions are framed using a modified version of Webb's (2007) depth of knowledge framework. Level 1 questions consist of items that require students to produce responses by following a set of rote procedures demonstrating procedural skills or recalling information (N. L. Webb, 2002). Level 2 questions are either conceptual or relate to problem-solving. Problem-solving items require students to produce responses to a given situation for which there can be a variety of possible solutions or approaches. Solution methods are not readily apparent and need students to decide how the problem can be solved (Porter, 2002). Conceptual-type items require students to respond by creating models and diagrams or demonstrating an understanding of mathematical properties and their applications (N. L. Webb, 2002). Level 3 items necessitate students to justify their reasoning or critique the reasoning of others through careful analysis and by explaining and modeling how and why a response is either correct or incorrect. Level 3 itemsmay also require models and diagrams to support responses (de Lange, 1999). For the example task, the module lists possible questions to ask students:

- Level 1 questions: *Explain the steps you followed to solve the problem. What do 4 and 8 in one-fourth and one-eighth tell you about the fractions?*
- Level 2 questions: *How does the number line model help explain the equation?*
- Level 3 questions: *Explain why Courtney is incorrect in her thinking. Mathematically, why might Courtney have made her comment?*

The models are presented in the typical order from either more informal to formal and from enactive to iconic to symbolic. The facilitation pattern initially focuses on describing the solution path depicted by the model, followed by establishing connections with other models when applicable. Each day, a different set of three students are asked to present their models and answer questions during the whole-class discussion.

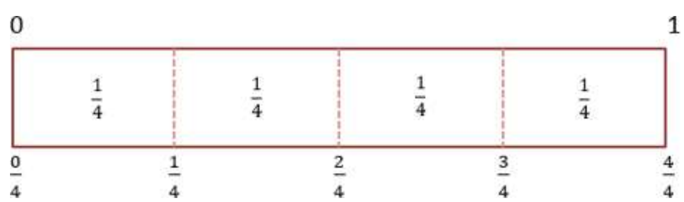
VARIED PRACTICE

Notably, missing in most curricula are practices for (a) articulating the process both verbally and in writing and (b) modeling through iconic and symbolic models and contexts. This enables all students' critical experiences to expand their thinking cognitively and progressively formalize their mathematical understanding (Freudenthal, 1991; Moschkovich, 2012). At this point in the lesson and throughout the module, students are formally asked to describe their processes and others' by focusing on the structure of mathematics. With regard to fractions, students must understand what units are and how they can be decomposed, composed, partitioned, and iterated to form newly relatedor equivalent units.

After the initial opening task, individual and small group work time, and the whole-class discussion, the module takes students through modeling, articulating ideas, and differentiated practice. Here is an example:

- *Enactive Model: Take a paper strip (rectangle) and fold it to make four equal parts. Label each section as ¼.*
- *Iconic Model: Draw a bar model (rectangle) and label it to represent 0, 1, and each ¼ section.*
- *Articulation: Describe each part of your bar model (Figure 3) with your neighbor using as many mathematical words that have been discussed so far in the module (e.g., unit, count, unit measure or size, iterate, partition, decompose, compose, etc.).*

Figure 3: Iconic Bar Model Representing Fourths



Next, students are asked to discuss the following questions with a partner and writetheir responses in sentence form.

Verbal and written form:

- *What does the fraction* $\frac{1}{4}$ *mean?*
- *What about* $\frac{3}{4}$*?*
- *What about* $\frac{5}{5}$*?*

This offers students the chance to use their own words first, usually informally, before being askedto articulate them more formally. Finally, students are given examples of the more formal written responses. For example:

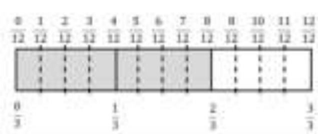
- *It takes 4 (*$\frac{1}{4}$ *units) to make 1 and you have counted only 1 of these* $\frac{1}{4}$ *units.*

- *It takes 4 ($\frac{1}{4}$ units) to make 1 and you have counted 3 of these $\frac{1}{4}$ units.*
- *It takes 4 ($\frac{1}{4}$ units) to make 1 and you have counted 5 of these $\frac{1}{4}$ units. That means $\frac{5}{4}$ is greater than 1.*

Given this feedback, students are asked to read each sentence aloud and then edit their written description to fit the newer, more formal form. Students should repeat this process with other fractions but with differentiation at this point in the module. Sets of different fractions that are easier,slightly more complex, and extended are used during the portion of the lesson.

Students move to differentiated practice after 2–4 lessons (typically 60 minutes each). Here, students are given a 3x3 matrix with four tasks in each column (see Figure 4). Students must solve four progressively more complex story problems using iconic and symbolic models to build fluency and flexibility. Given four progressively harder iconic models, students must then write a story problem that matches the visual model and provide a symbolic representation that matches the visual representation. Finally, given four progressively harder symbolic representations, students must create a context and visual model for the equation or expressions. The teacher is advised to have students solve two of the four tasks from each section.

Figure 4: Differential Practice Tasks

	Context	Visual (bar model)	Symbolic
Tasks 1 – 4	I only have a $\frac{1}{8}$ measuring cup. The recipe asks for a $\frac{1}{4}$ cup. How many $\frac{1}{8}$ measuring cups do I need to use?		
Tasks 5 – 8			
Tasks 9 – 12			$\frac{3}{5}=\frac{9}{15}$

SUMMARY OF DMT'S INSTRUCTION

By continuing this process over a 3–4-week period, the goal is to create enough situations for students to become better problem-solvers, make connections within and outside of mathematics, usemultiple mathematical models, and reason mathematically (Hiebert & Carpenter, 1992; Hiebert et al., 1997; M. A. Simon, 2006). The module framework develops mathematical thinking by providinghistorical or cultural problem-solving situations, space to articulate mathematical concepts and differentiated practice with the contexts and mathematical models.

REFERENCES

Ball, D. L., Hill, H. C., & Bass, H. (2005). Knowing mathematics for teaching: Who knows mathematics well enough to teach third grade, and how can we decide? *American Educator,*14-46.

Barnett-Clarke, C., Fisher, W., Marks, R., & Sharon, R. (2011). *Developing essential understanding of rational numbers for teaching mathematics in grades 3-5.:.* Reston, VA: National Councilof Teachers of Mathematics.

Borasi, R. (1987). Exploring mathematics through the analysis of errors. *For the Learning of Mathematics, 7*(3), 2-8.

Borasi, R. (1994). Capitalizing on errors as "Springboards for inquiry": A teaching experiment.

Journal for Research in Mathematics Education, 25(2), 166-208.

Brendefur, J. L., & Frykholm, J. A. (2000). Promoting mathematical communication in the classroom: Two preservice teachers' conceptions and practices. *Journal of MathematicsTeacher Education, 3*(2), 125-153.

Brendefur, J. L., Thiede, K., Strother, S., Bunning, K., & Peck, D. (2013). Developing mathematicalthinking: Changing teachers' knowledge and instruction. *Journal of Curriculum and Teaching, 2*(2).

Brendefur, J. L., Thiede, K., Strother, S., Jesse, D., & Sutton, J. (2016). A longitudinal study on the effects of professional development on elementary students' mathematics achievement.

Journal of Curriculum and Teaching, 5(2), 95-108. doi:10.5430

Bruner, J. S. (1964). The course of cognitive growth. *American Psychologist, 19*(1), 1-15. Bruner, J. S. (1996). *The culture of education*. Cambridge, Mass: Harvard University Press.

Cai, J., & Nie, B. (2007). Problem-solving in Chinese mathematics education: Research and practice.

ZDM, 39(5-6), 459-473.

Cai, J., & Silver, E. A. (1995). Solution processes and interpretations of solutions in solving a division-with-remainder story problem: Do Chinese and U.S. students have similar difficulties? *Journal for Research in Mathematics Education, 26*(5), 491-497.

Carpenter, T. P., Fennema, E., Peterson, P. L., Chiange, C. P., & Loef, M. (1989). Using knowledgeof children's mathematics thinking in classroom teaching: An experimental study. *American Educational Research Journal, 26*(4), 499-531.

Carpenter, T. P., Franke, M. L., Jacobs, V. R., & Fennema, E. (1998). A longitudinal study of invention and understanding in children's multidigit addition and subtraction. *Journal forResearch in Mathematics Education, 29*, 3-20.

Carpenter, T. P., & Lehrer, R. (1999). Teaching and learning mathematics with understanding. In E.Fennema & T. Romberg (Eds.), *Mathematics Classrooms that Promote Teaching for Understanding* (pp. 19 - 32). Mahwah, NJ: Lawerance Erlbaum Associates.

Chernoff, J. J., Flanagan, K. D., McPhee, C., & Park, J. (2007). Preschool: First Findings from the Preschool Follow-Up of the Early Childhood Longitudinal Study, BirthCohort (ECLS-B). First Look. NCES 2008-025. *National Center for Education Statistics*.

Clements, Xiufeng, L., & Sarama, J. (2008). Development of a measure of early mathematics achievement using the Rasch model: The Research-Based Early Maths Assessment. *Educational Psychology, 28*(4), 457-482.

Clements, D., & Sarama, J. (2007). Effects of a preschool mathematics curriculum: Summative research on the Building Blocks project. *Journal for Research in Mathematics Education, 38*,136-163. doi:doi.org/10.2307/30034954

Cobb, P. (2000). Conducting classroom teaching experiments in collaboration with teachers. In R. Lesh & A. Kelly (Eds.), *Handbook of research design in mathematics and science education*(pp. 307-334). Mahwah, NJ: Lawrence Erlbaum.

de Lange, J. (1999). *Framework for classroom assessment in mathematics*. Unpublished manuscript. Madison, WI.

Demmert, W. G., & Towner, J. C. (2003). A Review of the Research Literature on the Influences of Culturally Based Education on the Academic Performance of Native American Students. Final Paper.

Desimone, L. M. (2009). Improving impact studies of teachers' professional development: Toward better conceptualizations and measures. *Educational Researcher, 38*(3), 181-199.

Desimone, L. M., Porter, A. C., Garet, M. S., Yoon, K. S., & Birman, B. F. (2002). Effects of professional development on teachers' instruction: Results from a three-year longitudinalstudy. *Educational Evaluation and Policy Analysis, 24*(2), 81-112.

Doabler, C. T., Strand Cary, M., Jungjohann, K., Clarke, B., Fien, H., Baker, S., . . . Chard, D. (2012). Enhancing core mathematics instruction for students at risk for mathematics disabilities. *Teaching Exceptional Children, 44*(4), 48-57.

Doerr, H. M. (2006). Examining the tasks of teaching when using students' mathematical thinking. *Educational Studies in Mathematics, 62*(1), 3-24.

Dolk, M., & Fosnot, C. T. (2006). Minilessons for Operations with Fractions, Decimals, And Precents Addition and Subtraction: Heinemann.

Dossey, J. A. (1992). The nature of mathematics: Its role and its influence. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 29-48). New York, NY: Macmillan.

Driscoll, M. P. (2004). *Psychology of learning for instruction* (3rd ed.). Needham Heights, MA.: Allyn & Bacon.

Duncan, G. J., Dowsett, C. J., Claessens, A., Magnuson, K., Huston, A. C., Klebanov, P., . . . Japel,

C. (2007). School readiness and later achievement. *Dev Psychol, 43*(6), 1428-1446. doi:10.1037/0012-1649.43.6.1428

Fennema, E., Carpenter, T. P., Franke, M. L., Levi, L., Jacobs, V., & Emson, S. (1996). A longitudinal study of learning to use children's thinking in mathematics instruction. *Journalfor Research in Mathematics Education, 27*(4), 404-434.

Freudenthal, H. (1973). *Mathematics as an educational task*: Springer.

Freudenthal, H. (1991). Revisiting Mathematics Education: China Lectures.

Gersten, R., Beckmann, S., Clarke, B., Foegen, A., Marsh, L., Star, J. R., & Witzel, B. (2009).

Assisting students struggling with Mathematics: Response to intervention (RtI) for elementary and middle schools (NCEE 2009- 4060) Washington, DC: Institute of EducationSciences.

Ginsburg, H., Lee, J. S., & Boyd, J. S. (2008). Mathematics education for young children: What it is and how to promote it? *Social Policy Report, 22*(1).

Gravemeijer, K., & van Galen, F. (2003). Facts and algorithms as products of students' mathematical activity. In J. Kilpatrick, W. G. Martin, & D. Schifter (Eds.), *A research companion to principles and standards for school mathematics* (pp. 114-122). Reston, VA:NCTM.

Hiebert, J., & Carpenter, T. P. (1992). Learning and teaching with understanding. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 65-97). New York: Macmillan.

Hiebert, J., Carpenter, T. P., Fennema, E., Fuson, K., Wearne, D., & Murray, H. (1997). *Making sense: Teaching and learning mathematics with understanding*. Portsmouth, NH: Heinemann.

Hiebert, J., Carpenter, T. P., Fennema, E., Fuson, K. C., Human, P., Murray, H., . . . Wearne, D. (1996). Problem-solving as a basis for reform in curriculum and instruction: The case of mathematics. *Educational Researcher, 25*(4), 12-21.

Hill, H. C., Sleep, L., Lewis, J. M., & Ball, D. L. (2007). Assessing teachers' mathematical knowledge: What knowledge matters and what evidence counts? In F. Lester (Ed.),*Handbook for research on mathematics education* New York, NY: Routledge.

Jordan, N. C., Kaplan, D., Ramineni, C., & Locuniak, M. N. (2009). Early math matters: kindergarten number competence and later mathematics outcomes. *Dev Psychol, 45*(3), 850-867. doi:10.1037/a0014939

Larsen, S., & Bartlo, J. (2009). The role of tasks in promoting discourse supporting mathematical learning. In L. Knott (Ed.), *The role of mathematics discourse in producing leaders of discourse* (pp. 77-98). Charlotte, NC: Information Age Publishing.

Leinwand, S., & Ginsburg, A. L. (2007). Learning from Singapore math. *Educational Leadership, 65*(3), 32-36.

Lemke, M., Sen, A., Pahike, E., Partelow, L., Miller, D., & Williams, T. (2004). International outcomes of learning in mathematics literacy and problem-solving: Pisa 2003 results fromthe U.S. perspective. Retrieved from Washington DC

Lipka, J., & Adams, B. (2004). Culturally Based Math Education as a Way to Improve AlaskaNative Students' Math Performance. *Appalachian Collaborative Center for Learning*.

Mitchell, J. H., Hawkins, E. F., Stancavage, F. B., & Dossey, J. A. (2000). *Estimation skills, mathematics-in-context, and advanced skills in mathematics*. Retrieved from

Morgan, P. L., Farkas, G., & Qiong, W. (2009). Five-year growth trajectories of kindergarten children with learning difficulties in mathematics. *J Learn Disabil, 42*(4), 306-321. doi:10.1177/0022219408331037

Moschkovich, J. (1999). Supporting the participation of English Language Learners in mathematical discussions *For the Learning of Mathematics, 19*(1), 11-19.

Moschkovich, J. (2012). Mathematics, the Common Core, and language: Recommendations for mathematics instruction for ELs aligned with the Common Core. *Understanding language:*

Commissioned papers on language and literacy issues in the Common Core State Standardsand Next Generation Science Standards, 17-31.

Mullis, I. V. S., Martin, M. O., Gonzalez, E. J., & Chrostowski, S. J. (2004). *TIMSS 2003 international mathematics report.* Retrieved from Boston, MA:

NCES. (2013). The Nation's Report Card: A first look: 2013 Mathematics and Reading (NCES 2014-451). . Washington, D.C.: U.S. Department of Education.

NCES. (2015). *National Indian Education Study 2015.* Retrieved from Washington DC:

NCTM. (1989). *Curriculum and evaluation standards for school mathematics.* Reston, VA: National Council of Teachers of Mathematics.

NCTM. (1991). *Professional standards for the teaching of mathematics.* Reston, VA: National Council of Teachers of Mathematics.

NCTM. (2000). *Principles and standards for school mathematics.* Reston, VA: National Council of Teachers of Mathematics.

Newmann, F. M., & Associates. (1996). *Authentic achievement: Restructuring schools forintellectual quality.* San Francisco, CA: Jossey Bass.

NGA. (2010). *Common core state standards for mathematics.* Washington DC: National Governors Association and the Council of Chief State School Officers.

NMAP. (2008). *Foundations for success: The final report of the National Mathematics Advisory Panel.* Washington D.C.: U.S. Department of Education, Office of Planning, Evaluation andPolicy Development.

NRC. (2001). *Adding it up: Helping children learn mathematics.* Washington DC: National Academy Press.

NRC. (2009). Mathematics learning in early childhood: Paths toward excellence and equity.

Washington, DC: The National Academies Press.

Obara, S., & Sloan, M. (2010). Classroom experiences with new curriculum materials during the implementation of performance standards in mathematics: A case study of teachers copingwith change. *International Journal of Science and Mathematics Education, 8*(2), 349-372.

Pitkethly, A., & Hunting, R. (1996). A review of recent research in the area of initial fraction concepts. *Educational Studies in Mathematics, 30*(1), 5-38.

Reese, C. M., Miller, K. E., Mazzeo, J., & Dossey, J. A. (1997). *NAEP 1996 Mathematics Report Card for the Nation and the States.* Retrieved from Washington, DC:

Richland, L. E., Stigler, J. W., & Holyoak, K. J. (2012). Teaching the conceptual structure of mathematics. *Educational psychologist, 47*(3), 189-203.

Romberg, T. A., & Kaput, J. (1999). *Mathematics worth teaching, mathematics worthunderstanding.* Mahwah, NJ: Lawrence Erlbaum Associates.

Schmidt, W. H. (2012). At the precipice: The story of mathematics education in the United States. *Peabody Journal of Education, 87*(1), 133-156.

Schmidt, W. H., & Burroughs, N. A. (2013). Springing to Life. *American Educator*, 3.

Senk, S. L., & Thompson, D. R. (2003). Standards-based school mathematics curricula: What are they? What do students learn? : Lawrence Erlbaum Associates.

Siegler, R. S., & Alabali, M. (2004). *Children's thinking* (4th ed.): Prentice-Hall.

Simon, M., & Tzur, R. (2004). Explicating the role of mathematical tasks in conceptual learning: An elaboration of the hypothetical learning trajectory. *Mathematical Thinking and Learning, 6*(2), 91-104.

Simon, M. A. (2006). Key developmental understandings in mathematics: A direction for investigating and establishing learning goals. *Mathematical Thinking and Learning, 8*(4),359-371.

Stein, M. K., Engle, R. A., Smith, M. S., & Hughes, E. K. (2008). Orchestrating productive mathematical discussions: Five practices for helping teachers move beyond show and tell. *Mathematical Thinking and Learning, 10*(4), 313-340.

Trafton, P. R., Reys, B. J., & Wasman, D. G. (2001). Standards-based mathematics curriculum materials: A phrase In search of a definition. *Phi Delta Kappan, 83*(3), 259-264.

Treffers, A. (1987). Three dimensions: A model of goal and theory description in mathematics instruction – The Wiskobas Project. Reidel: Dordrecht, The Netherlands.

Watanabe, T. (2015). Visual Reasoning Tools in Action. *Mathematics Teaching in the Middle School, 21*(3), 152-160.

Webb, N. (2007). Issues related to judging the alignment of curriculum standards and assessments. *Applied Measurement in Education, 20*(1), 7-25.

Webb, N. L. (2002). Depth-of-knowledge levels for four content areas. *Language Arts, 28*(March).